Shocks and acceleration waves in moderm continuum mechanics and in social systems

B. Straughan*
Department of Mathematics
University of Durham, DH1 3LE, U.K.

November 8, 2013

Abstract

The use of discontinuity surface propagation (e.g. shock waves and acceleration waves) is well known in modern continuum mechanics and yields a very useful means to obtain important information about a fully nonlinear theory with no approximation whatsoever. A brief review of some of the recent uses of such discontinuity surfaces is given and then we mention modelling of some social problems where the same mathematical techniques may be used to great effect. We specifically show how to develop and analyse models for evolution of one language overtaking use of another leading to possible extinction of the former language. Then we analyse shock transmission in a model for the evolutionary transition from the human period when hunter-gatherers transformed into farming. Finally we address modelling discontinuity waves in the context of diffusion of an innovation.

1 Introduction

The topics of acceleration wave motion and their transition into shock waves is one with a rich history. Such discontinuity surface analyses have provided a means to obtaining highly useful information on the transmission of a wave in the fully nonlinear situation without any approximation whatsoever. Largely because of this fact, use of such discontinuity surface waves is still very prevalent in the modern continuum mechanics literature, see e.g. Bissell & Straughan [6], Christov & Jordan [8, 9, 10], Christov *et al.* [11, 12], Currò *et al.* [15, 16], Fabrizio *et al.* [18], Fabrizio & Morro [19], Gentile & Straughan [21], Jordan [26, 27, 28, 29, 30, 31, 32], Marasco [39, 40], Marasco & Romano [41], Paoletti

*Dedicated to Mauro Fabrizio with great admiration and affection. Research supported by a Research Grant of the Leverhulme Trust, “Tippings Points: Mathematics, Metaphors and Meanings”. I am indebted to two anonymous referees for carefully reading the original manuscript. Their comments have led to improvements in the paper.

[47], Ruggeri & Sugiyama [48], Sharma & Venkatramani [49], Straughan [50, 51, 52], and Valenti *et al.* [55]. In particular, hyperbolic theories in continuum mechanics have recently been shown to be dominant in many cases rather than traditional diffusion mechanisms, see e.g. Christov [7], Ciarletta *et al.* [14], Gentile & Straughan [22], Morro [42], and many such cases are reported in the book by Straughan [52].

In another development, mathematical modelling and analysis of social problems has became a dominant topic. With governments having to spend huge amounts of money to combat problems of alcoholism, drug abuse, obesity, and many others, there is clearly a need for mathematics to develop solutions via predictive modelling. Such models and their associated solutions are already appearing, e.g. in alcoholism, Huo & Song [24], Mulone & Straughan [44], Walters *et al.* [56]; in anorexia and in bulimia, Ciarcià *et al.* [13]; in heroin and other drug abuse, Kalula & Nyabadza [34], Liu & Zhang [38], Mulone & Straughan [43], Nyabadza *et al.* [45, 46]; in integration of different ethnic communities, Fabrizio [17]; in crowd behavioural problems, Bellomo & Bellouquid [3], Bellomo *et al.* [4], Bissell & Straughan [6]; in smoking, Bissell *et al.* [5]; in virus transmission, such as the Hantavirus or HIV, Barbera *et al.* [2], Bissell & Straughan [6], Hussaini & Winter [25]; and many further references to these challenging topics are provided in these articles.

Apart from stability studies and travelling wave analyses, another important way to tackle models for cultural and social problems was developed by Jordan [29]. In a novel piece of work Jordan [29] developed a detailed analysis for the evolutionary behaviour of a shock wave for a hyperbolic version of the famous Fisher equation

$$\frac{\partial \rho}{\partial t} - \nu \frac{\partial^2 \rho}{\partial x^2} = \gamma \rho \Big(1 - \frac{\rho}{\rho_s} \Big) \tag{1}$$

where $\rho(x,t)$ is a density and ν, γ and ρ_s are positive constants. This equation was proposed by Fisher [20] as a model for the spread of an advantageous gene, and was also simultaneously proposed by Kolmogoroff *et al.* [37]. The articles of Jordan [29] and of Jordan & Puri [33] refer to equation (1) as the Fisher - KPP equation. Jordan [29] pertinently points out that equation (1) has been studied with great effect in a variety of contexts in the biological, physical and social sciences.

Jordan [29], in fact, uses Green & Naghdi [23] thermodynamics (cf. Straughan [52]) to replace equation (1) by a pair of equations which convert it to a hyperbolic system. Jordan's [29] system is

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= - \frac{\partial q}{\partial x} + \gamma \rho \Big(1 - \frac{\rho}{\rho_s} \Big) \,, \\ \frac{\partial q}{\partial t} &= - V^2 \frac{\partial \rho}{\partial x} \,, \end{aligned} \tag{2}$$

where V is a constant which turns out to be the shock speed and q is the flux for the density ρ. Jordan's important ideas have recently been extended to study a nonlinear model for a mutant gene which is linked to a cultural trait by Straughan [53].

In this paper we firstly illustrate the mathematical theory of an acceleration wave by studying wave motion in an elastic body which has a double porosity structure. We stress, though, that both acceleration and shock wave techniques are useful as a means of obtaining information from models for social problems or for those of concern in a cultural context. This is a new application area for the theory of singular surfaces and hinges on Jordan's [29] fundamental paper. In this connection we develop hyperbolic models for three areas of interest. The first problem we consider is one of a language in a mixed language community becoming dominant with possible extinction of other languages. We then study a model for the transmission of human behaviour from the period where we lived by hunting and by gathering such foods as berries to that when mankind became organised farmers. The final problem we present analyses a model for the transmission of an innovation, often referred in the business and economic literature as diffusion of innovation. We show how to derive the equations for the propagation of a shock wave in each case. In each of the cases we derive the shock speed and then derive a coupled set of nonlinear ordinary differential equations for the amplitudes of the shock wave. These systems are easily integrated numerically.

2 Acceleration waves in elasticity with a double porosity structure

A background to the theory of elastic bodies with double porosity may be found in e.g. Straughan [54], see also the references therein. We simply present the relevant linearized equations and these are

$$\rho\ddot{u}_i =(a_{ijkh}u_{k,h})_{,j} - (\beta_{ij}p)_{,j} - (\gamma_{ij}q)_{,j}\,, \tag{3}$$

$$\alpha_1\dot{p} =(k_{ij}p_{,j})_{,i} - \hat{\gamma}(p-q) - \beta_{ij}\dot{u}_{i,j}\,, \tag{4}$$

$$\alpha_2\dot{q} =(m_{ij}q_{,j})_{,i} + \hat{\gamma}(p-q) - \gamma_{ij}\dot{u}_{i,j}\,, \tag{5}$$

where $\rho(\mathbf{x})$ is the solid density, $a_{ijkh}(\mathbf{x})$ are the elastic coefficients, $\beta_{ij}(\mathbf{x})$, $\gamma_{ij}(\mathbf{x})$ are constitutive coefficients which couple equations (3) - (5), $\alpha_1(\mathbf{x}) > 0$ and $\alpha_2(\mathbf{x}) > 0$ are measures of compressibilities of the macro pore and fissure systems, respectively, $k_{ij}(\mathbf{x})$, $m_{ij}(\mathbf{x})$ are anisotropic permeabilities, and $\hat{\gamma}(\mathbf{x})$ is an internal transport coefficient. In addition u_i is the elastic displacement, p is the macro pressure and q is the micro pressure. Throughout this article we employ standard indicial notation in conjunction with the Einstein summation convention. Hence, for example, a subscript $,j$ denotes $\partial/\partial x_j$, and a superposed dot denotes $\partial/\partial t$, namely, partial differentiation with respect to time.

The compatibility conditions for an acceleration wave analysis may be found in Fabrizio & Morro [19], and in Straughan [50], pp. 297–374, or Straughan [52], pp. 100–136. An acceleration wave for the system (3) - (5) is a two-dimensional surface, $\mathcal{S}$, in $\mathbb{R}^3$, such that $u_i(\mathbf{x},t)$, $p(\mathbf{x},t)$ and $q(\mathbf{x},t)$ are C^1 everywhere but across $\mathcal{S}$, $\ddot{u}_i, \dot{u}_{i,j}, u_{i,jk}$, $\ddot{p}, \dot{p}_{,i}, p_{,ij}$, $\ddot{q}, \dot{q}_{,i}, q_{,ij}$, and their higher derivatives suffer a finite discontinuity. The jump of a function f across $\mathcal{S}$, $[f]$, is defined as

$$[f] = f^- - f^+ \tag{6}$$

where

$$f^+ = \lim_{\mathbf{x}\to\mathcal{S}} f(\mathbf{x},t) \qquad \text{from the right,}$$

and

$$f^- = \lim_{\mathbf{x}\to\mathcal{S}} f(\mathbf{x},t) \qquad \text{from the left.}$$

We define the amplitudes $a_i(t), P(t), Q(t)$ of the acceleration wave as

$$a_i(t) = [\ddot{u}_i], \qquad P(t) = [\ddot{p}(t)], \qquad Q(t) = [\ddot{q}(t)]\,. \tag{7}$$

Upon taking the jumps of equations (3) - (5) one finds

$$\rho[\ddot{u}_i] = a_{ijkh}[u_{k,hj}] \tag{8}$$

together with

$$k_{ij}[p_{,ji}] = \beta_{ij}[\dot{u}_{i,j}] \qquad \text{and} \qquad m_{ij}[q_{,ji}] = \gamma_{ij}[\dot{u}_{i,j}]. \tag{9}$$

If U_N denotes the wavespeed then one has the Hadamard relation

$$\frac{\delta}{\delta t}[f] = [\dot{f}] + U_N[f_{,i}n_i] \tag{10}$$

cf. Straughan [50], equation (7.27), the compatibility relations,

$$[f_{,i}] = n_i[f_{,j}n_j] \qquad \text{and} \qquad [f_{,ij}] = n_in_j[f_{,rs}n_rn_s]\,. \tag{11}$$

Given the regularity of an acceleration wave for system (3) - (5) we use the Hadamard relation (10) to deduce that

$$[\ddot{u}_i] = -U_N[n_a\dot{u}_{i,a}] = U_N^2[n_an_bu_{i,ab}] \tag{12}$$

and we employ the compatibility relation to see that

$$[u_{k,hj}] = n_hn_j[u_{k,rs}n_rn_s]\,. \tag{13}$$

Further use of the Hadamard relation allows us to deduce

$$[\ddot{p}] = U_N^2[n_in_jp_{,ij}], \qquad [\ddot{q}] = U_N^2[n_in_jq_{,ij}]\,. \tag{14}$$

Employing expressions (12) and (13) in equation (8) we find

$$(\rho U_N^2\delta_{ij} - Q_{ij})a_j = 0 \tag{15}$$

where Q_{ij} is the acoustic tensor

$$Q_{ij} = a_{irjs}n_rn_s\,. \tag{16}$$

If $a_i = a(t)\lambda_i$ then from (15) we see that an acceleration wave will propagate when $\boldsymbol{\lambda}$ is an eigenvector of $\mathbf{Q}$. Once one knows λ_i then the wavespeed U_N follows from (15) as

$$U_N^2 = \frac{Q_{ij}\lambda_i\lambda_j}{\rho}\,. \tag{17}$$

If $\lambda_i = n_i$, where $\mathbf{n}$ is the unit normal to the wave $\mathcal{S}$, then the acceleration wave is longitudinal.

Employing (9) together with the compatibility relations we then find

$$k_{ij}n_in_jP = -\frac{\beta_{ij}n_j}{U_N}\,a_i \qquad \text{and} \qquad m_{ij}n_in_jQ = -\frac{\gamma_{ij}n_j}{U_N}\,a_i\,. \tag{18}$$

Thus, once we know a_i we may determine the wave amplitudes P and Q from equations (18).

In fact, one can proceed and determine the wave amplitude a_i by differentiation of (3). The analysis is more transparent if we consider a plane wave since this effectively reduces to the one-dimensional situation, to which we now restrict attention.

2.1 Wave amplitudes in one space dimension

Let now A, β, γ, k and m be the one-dimensional equivalents of the tensors $a_{ijkh}, \beta_{ij}, \gamma_{ij}, k_{ij}$ and m_{ij} and for simplicity we suppose these are constants. Then the one-dimensional forms of equations (3) - (5) become

$$\begin{aligned} \rho\ddot{u} &= Au_{xx} - \beta p_x - \gamma q_x \\ \alpha_1\dot{p} &= kp_{xx} - \hat{\gamma}(p-q) - \beta\dot{u}_x \\ \alpha_2\dot{q} &= mq_{xx} + \hat{\gamma}(p-q) - \gamma\dot{u}_x \end{aligned} \tag{19}$$

where $p_x = \partial p/\partial x$, etc. The one-dimensional wavespeed, V, satisfies $V^2 = A/\rho$ and equations (18) now reduce to

$$kP = -\frac{\beta a}{V}\,, \qquad mQ = -\frac{\gamma a}{V}\,. \tag{20}$$

If we differentiate equation $(19)_1$ with respect to x and take the jumps we then find

$$\rho[\ddot{u}_x] = A[u_{xxx}] - \beta[p_{xx}] - \gamma[q_{xx}]\,. \tag{21}$$

From the Hadamard relation we have

$$[\ddot{u}] = -V[\dot{u}_x] = V^2[u_{xx}]\,, \tag{22}$$

and

$$\frac{\delta}{\delta t}[\dot{u}_x] = [\ddot{u}_x] + V[\dot{u}_{xx}]\,, \qquad \frac{\delta}{\delta t}[u_{xx}] = [\dot{u}_{xx}] + V[u_{xxx}]\,. \tag{23}$$

Combining (23) we find

$$[\ddot{u}_x] = -\frac{2}{V}\,\frac{\delta a}{\delta t} + V^2[u_{xxx}]\,,$$

and use of this in (21) shows that

$$-\frac{2\rho}{V}\frac{\delta a}{\delta t}+\rho V^2[u_{xxx}]=A[u_{xxx}]-\frac{\beta P}{V^2}-\frac{\gamma Q}{V^2}\,. \tag{24}$$

Thus, using the fact that $V^2 = A/\rho$ and equations (20) we find equation (24) reduces to

$$\frac{\delta a}{\delta t}=-\frac{1}{2\rho V^2}\left(\frac{\beta^2}{k}+\frac{\gamma^2}{m}\right)a. \tag{25}$$

This, of course, integrates to show the wave amplitude decays exponentially. This is becasue we started with the linearized equations (3) - (5) or (19). For the fully nonlinear case the situation is different, for example one may witness finite time blow-up of the amplitude, see Gentile & Straughan [21].

3 Language extinction

The question of language extinction is an important one and mathematical models for this are proposed by Kandler & Steele [35] and by Kandler *et al.* [36]. They use as an example the fact that while Scottish Gaelic was the dominant language in the islands off North West Scotland, and in the North West of Scotland itself in the late 1800's, the English language is slowly taking over as the dominant language there. There are many examples of such language competition worldwide including such as the old Neapolitan language of Naples and its competition with Italian.

Kandler & Steele [35] develop a two component model for language competition. They suppose $v(\mathbf{x},t)$ represents the number of speakers of a language $\mathcal{B}$ in a region $\Omega \subseteq \mathbb{R}^2$, say, while $u(\mathbf{x},t)$ represents the number of speakers of a different language $\mathcal{A}$ which threatens to overtake language $\mathcal{B}$ due to the movement of speakers of language $\mathcal{A}$ into the region Ω. Their model is based on the equations

$$\begin{aligned}\frac{\partial u}{\partial t}&=d_1\Delta u+a_1u-b_1u^2+c_1uv,\\ \frac{\partial v}{\partial t}&=d_2\Delta v+\alpha_1 v-\beta_1v^2-c_1uv,\end{aligned} \tag{26}$$

where $d_1, d_2, a_1, b_1, c_1, \alpha_1$ and β_1 are positive constants. It is convenient to non-dimensionalize equations (26) and so let $\mathcal{T} = L^2/d_1$ be a time scale with $L = \sqrt{d_1/c_1}$ being the length scale. Let now $\hat{d} = d_1/d_2$, $a = a_1/c_1, b = b_1/c_1, \alpha = \alpha_1\hat{d}/c_1$, $\beta = \beta_1\hat{d}/c_1$, and then equations (26) may be written in the non-dimensional form

$$\begin{aligned}\frac{\partial u}{\partial t}&=\Delta u+au-bu^2+uv,\\ \hat{d}\,\frac{\partial v}{\partial t}&=\Delta v+\alpha v-\beta v^2-\hat{d}\,uv.\end{aligned} \tag{27}$$

We wish to study propagation of a shock wave via a hyperbolic form of equations (27), i.e. we wish to study how effectively a sudden influx of speakers of language $\mathcal{A}$ into Ω would propagate and influence the numbers u and v. To this end we generalize equations (27) into a hyperbolic form in the manner of the method of Jordan [29], as was done for gene-culture waves by Straughan [53]. Thus, let J and R be fluxes associated to u and v, respectively. Then take $\hat{d} = 1$, assume Ω is in one-dimension, and write instead of equations (27) the system

$$\begin{aligned} &u_t = -J_x + au - bu^2 + uv, \\ &\tau J_t + J = -u_x, \\ &v_t = -R_x + \alpha v - \beta v^2 - uv, \\ &\tau R_t + R = -v_x. \end{aligned} \tag{28}$$

To understand the above procedure of how and why we change the parabolic form into a hyperbolic one consider equation $(27)_1$ in one space dimension. Equation $(27)_1$ is essentially employing a flux simultaneously with Fourier's law, i.e. equation $(27)_1$ is

$$u_t = -J_x + au - bu^2 + uv$$

where the flux has the Fourier form

$$J = -\frac{\partial u}{\partial x}\,. \tag{29}$$

Jordan's [29] idea (for the Fisher equation) was to replace the Fourier law (29) by a Cattaneo one of form

$$\tau\frac{\partial J}{\partial t} + J = -\frac{\partial u}{\partial x}\,, \tag{30}$$

where $\tau > 0$ is a relaxation time. A detailed explanation of Cattaneo's equations and other thermodynamic laws, and why they are needed, including a historical account, is given in the book of Straughan [52]. We stress that hyperbolic forms of equations for many sociological or biological systems have been the subject of recent study, e.g. in traffic flow, in population dynamics, in fish migration, in the spread of a virus, in chemotaxis, and in skin burns, see chapter 9 of Straughan [52]. For the history and motivation of such hyperbolic systems as (28) we refer to Straughan [52] and we here concentrate on an analysis of shock waves in this system.

A shock wave for system (28) is a singular surface $\mathcal{S}$ such that u, v, J and R suffer a finite discontinuity on $\mathcal{S}$ but are at least C^1 everywhere else. If the shock speed is V, the Rankine-Hugoniot equations from (28) are

$$\begin{aligned} &V[u] = [J], \qquad \tau V[J] = [u], \\ &V[v] = [R], \qquad \tau V[R] = [v], \end{aligned} \tag{31}$$

cf. Straughan [53], equations (6). The novel aspect of Jordan's [29] work was to show that a shock wave for the hyperbolic Fisher system propagates in a sense in the manner of an acceleration wave in many continuum mechanics systems and leads to a Bernoulli equation for the shock amplitude. In our case we do not arrive at a single Bernoulli equation since (28) is a system of four equations. Nevertheless, we proceed by now taking the jump of equations (28).

Note that the Rankine-Hugoniot equations (31) show

$$V^2 = \frac{1}{\tau} > 0, \tag{32}$$

so V is constant. The Hadamard relation shows that

$$\frac{\delta}{\delta t}\,[u] = [u_t] + V[u_x], \tag{33}$$

and we repeatedly employ this together with the analogous form for v. Define the shock ampltudes $A(t), B(t), C(t)$ and $D(t)$ by

$$A = [u], \qquad B = [v], \qquad C = [J], \qquad D = [R].$$

Then, from the jumps of (28) and use of the Hadamard relation (33) we may derive the equations

$$\dot{A} - V[u_x] = -[J_x] + aA - bA^2 - 2bu^+A + AB + v^+A + u^+B, \tag{34}$$

$$\tau(\dot{C} - V[J_x]) + C = -[u_x], \tag{35}$$

$$\dot{B} - V[v_x] = -[R_x] + \alpha B - \beta B^2 - 2\beta v^+B - AB - v^+A - u^+B, \tag{36}$$

and

$$\tau(\dot{D} - V[R_x]) + D = -[v_x], \tag{37}$$

where here a superposed dot denotes $\delta/\delta t$, e.g. $\dot{A} = \delta A/\delta t$, and we have employed the relation for the product of a jump

$$[fg] = f^+[g] + g^+[f] + [f]\,[g]\,. \tag{38}$$

We next use the fact that V is constant and from (31), $VA = C$, $VB = D$, and we then form the combinations $V\times(34)+\tau^{-1}\times(35)$ and $V\times(36)+\tau^{-1}\times(37)$. In this way one arrives at the coupled system of nonlinear ordinary differential equations for the wave amplitudes A and B, of form

$$\begin{aligned} 2\dot{A} &= \Big(a - 2bu^+ + v^+ - \frac{1}{\tau}\Big)A + u^+B - bA^2 + AB, \\ 2\dot{B} &= \Big(\alpha - 2\beta v^+ - u^+ - \frac{1}{\tau}\Big)B - v^+A - \beta B^2 - AB. \end{aligned} \tag{39}$$

The evolution of the shock amplitude may be determined for given u^+, v^+ by solving equations (39) numerically, cf. the numerical solutions for the gene-culture system in Straughan [53].

A particular situation of interest might be when $u^+ = 0$. This could, for example, correspond to the situation where a group of English speakers arrives on an island on which there are only Gaelic speakers. In this case system (39) reduces to the simplified form

$$\begin{aligned} 2\dot{A} &= \lambda_1 A - bA^2 + AB, \\ 2\dot{B} &= \lambda_2 B - v^+ A - \beta B^2 - AB, \end{aligned} \tag{40}$$

where the coefficients λ_1 and λ_2 are given by

$$\lambda_1 = a + v^+ - \frac{1}{\tau} \qquad \text{and} \qquad \lambda_2 = \alpha - 2\beta v^+ - \frac{1}{\tau}\,.$$

Steady states of system (40) are easily found by setting the left hand sides of equations (40) to be equal to zero. One then finds there are four plausible steady states $(\bar{A}, \bar{B})$ given by

$$\begin{array}{llll} (i) & (0,0) & (ii) & \Big(0, \dfrac{\lambda_2}{\beta}\Big) \\ (iii) & \Big(\dfrac{\lambda_1}{b} + \dfrac{B_1}{b}\,, B_1\Big) & (iv) & \Big(\dfrac{\lambda_1}{b} + \dfrac{B_2}{b}\,, B_2\Big) \end{array} \tag{41}$$

where B_1, B_2 are solutions to

$$B_{1,2} = \frac{\dfrac{-v^+ - \lambda_1}{b} + \lambda_2 \pm \sqrt{(\dfrac{v^+ + \lambda_1}{b} - \lambda_2)^2 - \dfrac{4\lambda_1 v^+}{b}(\beta + \dfrac{1}{b})}}{2(\beta + \dfrac{1}{b})}$$

One may study the stability of these steady states and analyse how a shock evolves to such, as is done in the gene-culture model by Straughan [53].

Kandler *et al.* [36] present a three component model for language competition. They let u and w be the numbers of speakers of the single languages A and B and they further let v be the number of people who speak both. Their model actually involves a logistic term which has a nonlinear coefficient of the u^2, v^2 and w^2 terms. If we adopt their model but retain the structure of the Kandler & Steele [35] model, with equal diffusion coefficients, then one may derive the following set of equations for the evolution of u, v and w:

$$\begin{aligned} u_t &= d\,u_{xx} + a_1 u(1 - ku) - c_{31}\,uw + c_{12}\,uv, \\ v_t &= d\,v_{xx} + a_2 v(1 - kv) + (c_{13} + c_{31})uw - c_{32}\,vw - c_{12}\,uv, \\ w_t &= d\,w_{xx} + a_3 w(1 - kw) - c_{13}\,uw + c_{32}\,vw. \end{aligned}$$

If one non-dimensionalizes this system by choosing time and length scales as $\mathcal{T} = L^2/d$, $L = \sqrt{d/c_{12}}$ and then one selects $\alpha = a_1/c_{12}$, $\delta = c_{31}/c_{12}$, $\beta = a_2/c_{12}$, $\mu = c_{32}/c_{12}$, $\gamma = a_3/c_{12}$, and $\epsilon = c_{13}/c_{12}$, then one may arrive at the following non-dimensional form of equations

$$\begin{aligned} u_t &= u_{xx} + \alpha u - \alpha k u^2 - \delta uw + uv, \\ v_t &= v_{xx} + \beta v - \beta k v^2 + (\epsilon + \delta)uw - uv - \mu vw, \\ w_t &= w_{xx} + \gamma w - \gamma k w^2 - \epsilon uw + \mu vw. \end{aligned} \tag{42}$$

By using the Jordan - Cattaneo method, Jordan [29], one easily derives a hyperbolic analogue of (42) employing J, K and L as the fluxes for u, v and w, to find

$$\begin{aligned}
&u_t = -J_x + \alpha u - \alpha k u^2 - \delta uw + uv,\\
&\tau J_t + J = -u_x,\\
&v_t = -K_x + \beta v - \beta k v^2 + (\epsilon + \delta) uw - uv - \mu vw,\\
&\tau K_t + K = -v_x,\\
&w_t = -L_x + \gamma w - \gamma k w^2 - \epsilon uw + \mu vw,\\
&\tau L_t + L = -w_x.
\end{aligned} \tag{43}$$

One could develop a shock wave analysis for system (43) and arrive at three coupled nonlinear ordinary differential equations for the wave amplitudes which would then need to be solved numerically. Details for a system of six equations describing the "diffusion of an innovation" are given in section 5.

4 Hunter - gatherer to farmer transition

A very interesting system of equations is derived by Aoki *et al.* [1]. They model the situation where the population involves farmers and hunter-gatherers. The idea is that some of the hunter-gatherers come into contact with farmers and are converted to farming. Aoki *et al.* [1] denote the densities of each of these categories of people by $F(x,t)$, farmers, $C(x,t)$, converted farmers, and $H(x,t)$, hunter-gatherers. They derive a non-dimensional form of their equations as

$$\begin{aligned}
&F_t = F_{xx} + aF(1 - F - C),\\
&C_t = C_{xx} + C(1 - F - C) + s(F + C)H,\\
&H_t = H_{xx} + bH(1 - H - g\{F + C\}),
\end{aligned} \tag{44}$$

for constants a, b, g and s. One could clearly write a hyperbolic analogue of system (44) and study shock wave evolution.

We here consider a simplified model in which $a = 1$ (this means the initial growth rate of farmers and converted farmers is the same). If we then let $A = F + C$ from equations (44) we may derive the system

$$\begin{aligned}
&H_t = H_{xx} + bH(1 - H - gA),\\
&A_t = A_{xx} + A(1 - A + sH).
\end{aligned} \tag{45}$$

If we follow the Jordan-Cattaneo approach then from (45) one may derive the following hyperbolic system of equations,

$$\begin{aligned}
&H_t = -J_x + bH(1 - H - gA),\\
&\tau J_t + J = -H_x,\\
&A_t = -M_x + A(1 - A + sH),\\
&\tau M_t + M = -A_x,
\end{aligned} \tag{46}$$

where τ is a positive constant and where J and M are the fluxes related to H and A.

If one defines the wave amplitudes h and a by $h(t) = [H]$ and $a(t) = [A]$ then one may develop a shock wave analysis as in section (3). We omit details, but one may show the wavespeed V is given by $V^2 = 1/\tau$, and the coupled system of amplitude equations is

$$\begin{aligned} &\frac{\delta h}{\delta t} + \frac{1}{2}\Big(\frac{1}{\tau} - b + 2bH^+ + bgA^+\Big)h + \frac{bg}{2}\,H^+a + \frac{b}{2}\,h^2 + \frac{bg}{2}\,ah = 0, \\ &\frac{\delta a}{\delta t} + \frac{1}{2}\Big(\frac{1}{\tau} - 1 + 2A^+ - sH^+\Big)a - \frac{sA^+}{2}\,h - \frac{s}{2}\,ah = 0. \end{aligned} \tag{47}$$

Numerical solutions of system (47) are found in a straightforward manner.

5 Diffusion of innovation shock waves

The final model we consider is one of economic and business interest and involves the diffusion of an innovation. Wang *et al.* [57] produce an ordinary differential equation model for the diffusion of an innovation which divides the relevant public who might buy this product into the class of people not aware of the product, N, those aware but who have not yet adopted the product, I, and those who have already adopted the product, A. If one denotes the total population (market potential) by m then $m = N + I + A$. The differential equations introduced by Wang *et al.* [57] are

$$\begin{aligned} \frac{dN}{dt} &= -pN - \frac{q_1}{m}\,AN + \gamma I + \mu A, \\ \frac{dI}{dt} &= pN + \frac{q_1}{m}\,AN - \gamma I - g\Big(\frac{A}{m}\Big)\,I, \\ \frac{dA}{dt} &= -\mu A + g\Big(\frac{A}{m}\Big)\,I, \end{aligned} \tag{48}$$

where p, q_1, γ and μ are positive constants, and g is a function, e.g. $g(x) = \alpha + c_1x + c_2x^2$, α, c_1, c_2 constants. We here concentrate on one of the functions chosen by Wang *et al.* [57], namely, $g = \alpha + c_1x$. Wang *et al.* [57] investigate the stability of solutions to the ordinary differential equation system (48).

Our interest is in studying how an innovation will move in both space and time and so we consider adding flux terms to each equation in (48) and allowing N, I and A to depend on both x and t. Let now J_N, J_I and J_A be fluxes associated to the variables N, I and A. Then for τ a relaxation time we employ the Jordan-Cattaneo argument, cf. Jordan [29], to generalize (48) into the system of equations

$$
\begin{aligned}
&\frac{\partial N}{\partial t} = -\frac{\partial J_N}{\partial x} - pN - \frac{q_1}{m} AN + \gamma I + \mu A, \\
&\tau\frac{\partial J_N}{\partial t} + J_N = -D\frac{\partial N}{\partial x}\,, \\
&\frac{\partial I}{\partial t} = -\frac{\partial J_I}{\partial x} + pN + \frac{q_1}{m} AN - \gamma I - \Big(\alpha + c_1\frac{A}{m}\Big)\, I, \\
&\tau\frac{\partial J_I}{\partial t} + J_I = -D\frac{\partial I}{\partial x}\,, \\
&\frac{\partial A}{\partial t} = -\frac{\partial J_A}{\partial x} - \mu A + \Big(\alpha + c_1\frac{A}{m}\Big)\, I, \\
&\tau\frac{\partial J_A}{\partial t} + J_A = -D\frac{\partial A}{\partial x}\,,
\end{aligned}
\tag{49}
$$

D being a positive constant, the diffusion coefficient.

We define a shock wave for equations (49) to be a singular surface, $\mathcal{S}$, such that N, I and A and their higher derivatives possess a discontinuity on $\mathcal{S}$, but are at least C^1 everywhere else.

We provide some details for the development of the equations for the shock amplitudes because the calculations are non-standard due to the presence of the $1/m$ terms. The Rankine-Hugoniot equations for (49) are shown to be

$$
\begin{aligned}
V[N] &= [J_N], \qquad \tau V[J_N] = D[N], \\
V[I] &= [J_I], \qquad \tau V[J_I] = D[I], \\
V[A] &= [J_A], \qquad \tau V[J_A] = D[A],
\end{aligned}
\tag{50}
$$

from which we find

$$
V^2 = \frac{D}{\tau}\; > 0, \tag{51}
$$

so that the shock speed V is a constant.

We define the shock amplitudes $E(t), F(t)$ and $G(t)$ to be

$$
E = [N], \qquad F = [I], \qquad G = [A], \tag{52}
$$

and then we take the jumps of each of the equations (49). The Hadamard and product rules are used extensively. For example, from $(49)_{1,2}$ we find

$$
\frac{\delta}{\delta t}\,[N] - V\Big[\frac{\partial N}{\partial x}\Big] = -\Big[\frac{\partial J_N}{\partial x}\Big] - p[N] - q_1\Big[\frac{AN}{m}\Big] + \gamma[I] + \mu[A], \tag{53}
$$

and

$$
\tau\frac{\delta}{\delta t}\,[J_N] - \tau V\Big[\frac{\partial J_N}{\partial x}\Big] + [J_N] = -D\Big[\frac{\partial N}{\partial x}\Big]\,. \tag{54}
$$

Now form (53)+$(\tau V)^{-1}$(54) to derive with the aid of the Rankine-Hugoniot equations (50),

$$
2\frac{\delta E}{\delta t} + \frac{1}{\tau}\,E = -pE - q_1\Big[\frac{AN}{m}\Big] + \gamma F + \mu G. \tag{55}
$$

To progress from this point we need to observe that with $J = J_N + J_I + J_A$, by adding appropriate groups of equations from (49) we obtain

$$\begin{aligned} \frac{\partial m}{\partial t} &= -\frac{\partial J}{\partial x} \\ \tau\frac{\partial J}{\partial t} + J &= -D\frac{\partial m}{\partial x}\,. \end{aligned} \tag{56}$$

The Rankine-Hugoniot equations for the system (56) are

$$V\tau[J] = D[m], \qquad V[m] = [J], \tag{57}$$

so that $V^2 = D/\tau$, as above. Then from the jumps of (56) together with the Hadamard relation we find

$$\begin{aligned} &\frac{\delta}{\delta t}[m] - V\Big[\frac{\partial m}{\partial x}\Big] + \Big[\frac{\partial J}{\partial x}\Big] = 0, \\ &\tau\Big(\frac{\delta}{\delta t}[J] - V\Big[\frac{\partial J}{\partial x}\Big]\Big) + [J] + D\Big[\frac{\partial m}{\partial x}\Big] = 0. \end{aligned} \tag{58}$$

By adding equations (58) appropriately and by using (57) we thus find

$$\frac{\delta}{\delta t}[m] + \frac{1}{2\tau}\,[m] = 0. \tag{59}$$

Thus we find from (59) that the total amplitude (jump) is

$$[m(t)] = [m(0)]e^{-t/2\tau}\,. \tag{60}$$

Recall $[f] = f^- - f^+$ so that

$$\Big[\frac{1}{f}\Big] = \frac{1}{f^-} - \frac{1}{f^+}$$

and this may be rearranged to deduce

$$\Big[\frac{1}{f}\Big] = -\frac{1}{f^+} + \frac{1}{(f^+ + [f])}\,. \tag{61}$$

Define now

$$Q(t) = m^+ + [m(0)]e^{-t/2\tau}, \tag{62}$$

where we recall $m^+ = N^+ + I^+ + A^+$. Then

$$\Big[\frac{AN}{m}\Big] = A^+N^+\Big[\frac{1}{m}\Big] + \frac{1}{m^+}[AN] + [AN]\Big[\frac{1}{m}\Big]\,. \tag{63}$$

We employ (61) and (62) in (63) to find

$$\Big[\frac{AN}{m}\Big] = A^+N^+\Big\{-\frac{1}{m^+} + \frac{1}{Q(t)}\Big\} + \frac{A^+[N] + N^+[A] + [N]\,[A]}{Q(t)}\,. \tag{64}$$

Next, we return to equation (55) and employ (64) to derive

$$
\begin{aligned}
2\frac{\delta E}{\delta t} =& - E\Big(\frac{1}{\tau} + p + \frac{A^+ q_1}{Q(t)}\Big) + \gamma F + G\Big(\mu - \frac{N^+ q_1}{Q(t)}\Big) \\
& - \frac{q_1}{Q(t)}\, EG - q_1 A^+ N^+ \Big\{ -\frac{1}{m^+} + \frac{1}{Q(t)} \Big\}.
\end{aligned} \tag{65}
$$

The calculations involving equations $(49)_{3,4}$ and $(49)_{5,6}$ are similar. Omitting details one finds

$$
\begin{aligned}
2\frac{\delta F}{\delta t} =& - F\Big(\frac{1}{\tau} + \gamma + \alpha - \frac{c_1 A^+}{Q}\Big) + \frac{G}{Q}(q_1 N^+ - c_1 I^+) \\
& + E\Big(p + \frac{A^+ q_1}{Q(t)}\Big) + \frac{q_1}{Q}\, EG - \frac{c_1}{Q}\, FG \\
& + A^+(q_1 N^+ - c_1 I^+)\Big\{ -\frac{1}{m^+} + \frac{1}{Q(t)} \Big\}
\end{aligned} \tag{66}
$$

and

$$
\begin{aligned}
2\frac{\delta G}{\delta t} =& \Big(\alpha + \frac{c_1 A^+}{Q}\Big)F + G\Big(-\mu - \frac{1}{\tau} + \frac{c_1 I^+}{Q}\Big) \\
& + c_1 A^+ I^+ \Big\{ -\frac{1}{m^+} + \frac{1}{Q(t)} \Big\} + \frac{c_1}{Q}\, FG.
\end{aligned} \tag{67}
$$

Equations (65) - (67) are a nonlinear system of coupled ordinary differential equations for the wave amplitudes E, F and G, and these may be solved by a suitable numerical method.

While it is straightforward to solve (65) - (67) by a numerical method we observe that from (60) we have

$$
E + F + G = [m(0)]e^{-t/2\tau} = m_0 e^{-t/2\tau} \tag{68}
$$

where m_0 is the data term indicated. Thus, if we wish we may write $E = m_0 e^{-t/2\tau} - F - G$ and eliminate E from equations (66) and (67) to find

$$
\begin{aligned}
\frac{\delta F}{\delta t} =& - \frac{F}{2}\Big(\frac{1}{\tau} + \gamma + \alpha + p + \frac{(q_1 - c_1)A^+}{Q}\Big) \\
& + \frac{G}{2}\Big(\frac{q_1 N^+ - c_1 I^+}{Q} - p - \frac{q_1 A^+}{Q} + \frac{q_1 m_0 e^{-t/2\tau}}{Q}\Big) \\
& + \frac{A^+}{2}(q_1 N^+ - c_1 I^+)\Big\{ -\frac{1}{m^+} + \frac{1}{Q(t)} \Big\} \\
& + \frac{1}{2}\Big(p + \frac{A^+ q_1}{Q(t)}\Big) m_0 e^{-t/2\tau} - \Big(\frac{c_1 + q_1}{2Q}\Big) FG - \frac{q_1}{2Q}\, G^2,
\end{aligned} \tag{69}
$$

and

$$
\begin{aligned}
\frac{\delta G}{\delta t} =& \frac{1}{2}\Big(\alpha + \frac{c_1 A^+}{Q}\Big)F + \frac{G}{2}\Big(-\mu - \frac{1}{\tau} + \frac{c_1 I^+}{Q}\Big) \\
& + \frac{c_1 A^+ I^+}{2}\Big\{ -\frac{1}{m^+} + \frac{1}{Q(t)} \Big\} + \frac{c_1}{2Q}\, FG.
\end{aligned} \tag{70}
$$

An alternative way to solving (65) - (67) is to integrate equations (69) and (70) numerically.

6 Conclusions

The purpose of this article is to show that the mathematical techniques associated with shock waves, acceleration waves, and other discontinuity surfaces are not restricted to only the fields of continuum mechanics such as elasticity, or fluid mechanics. They represent a valuable way to obtain information concerning models in many other fields such as anthropology, business, economics, or social sciences. We have chosen to illustrate this by deriving the wavespeeds and the equations governing the evolution of the wave amplitudes by employing examples of language competition, the emergence of farming in a world where only hunter-gathering existed, and the propagation of an innovation, typically in a business or economic situation. Nevertheless, there are many emerging interdisciplinary areas where the techniques of discontinuity surfaces may be applied with undoubtedly highly useful results.

A key to the extension of discontinuity waves to other areas rather than just to continuum mechanics was the highly influential paper of Jordan [29], who showed how one could develop a hyperbolic analogue of the famous Fisher equation which has been employed extensively in biology and in business. Further examples of hyperbolic models in a variety of interdisciplinary scenarios are described at length in chapter 9 of the book by Straughan [52]. We believe that the methods of discontinuity surface analysis will provide a very useful way to explore many of the exciting emerging areas of applied mathematics where nonlinear wave propagation with no approximation is desirable.

References

[1] K. Aoki, M. Shida, and N. Shigesada. Travelling wave solutions for the spread of farmers into a region occupied by hunter-gatherers. *Theoretical Population Biology*, 50:1–17, 1996.

[2] E. Barbera, C. Currò, and G. Valenti. A hyperbolic reaction diffusion model for the hantavirus infection. *Math. Meth. Appl. Science*, 31:481–499, 2002.

[3] N. Bellomo and A. Bellouquid. On the modelling of crowd dynamics, looking at the beautiful shapes of swarms. *Networks and Heterogeneous Media*, 6:383–399, 2011. Article 569.

[4] N. Bellomo, B. Piccoli, and A. Tosin. Modelling crowd dynamics from a complex system viewpoint. *Math. Models Meth. Appl. Science*, 22:1–20, 2012. Article 1230004.

[5] J. J. Bissell, C. C. S. Caiado, M. Goldstein, and B. Straughan. Compartmental modelling of social dynamics with generalised peer incidence. *Math. Models Meth. Appl. Science*, 40:1–1, 2013.

[6] J. J. Bissell and B. Straughan. Discontinuity waves as tipping points. Discrete and Continuous Dynamical Systems B, in the press, 2013.

[7] C. I. Christov. On frame indifferent formulation of the Maxwell - Cattaneo model of finite - speed heat conduction. *Mechanics Research Communications*, 36:481–486, 2009.

[8] I. Christov and P. M. Jordan. Shock bifurcation and emergence of diffusive solitons in a nonlinear wave equation with relaxation. *New Journal of Physics*, 10:043027–1–043027–13, 2008.

[9] I. Christov and P. M. Jordan. Shock and traveling wave phenomena on an externally damped, non-linear string. *Int. J. Nonlinear Mech.*, 44:511–519, 2009.

[10] I. Christov and P. M. Jordan. On the propagation of second - sound in nonlinear media: shock, acceleration and traveling wave results. *J. Thermal Stresses*, 33:1109–1135, 2010.

[11] I. Christov, P. M. Jordan, and C. I. Christov. Nonlinear acoustic propagation in homentropic perfect gases: a numerical study. *Physics Letters A*, 353:273–280, 2006.

[12] I. Christov, P. M. Jordan, and C. I. Christov. Modelling weakly nonlinear acoustic wave propagation. *Quart. Jl. Mech. Appl. Math.*, 60:473–495, 2007.

[13] C. Ciarcià, P. Falsaperla, A. Giacobbe, and G. Mulone. A mathematical model of anorexia and bulimia. Manuscript, 2013.

[14] M. Ciarletta, B. Straughan, and V. Tibullo. Christov-Morro theory for non-isothermal diffusion. *Nonlinear Analysis: Real World Applications*, 13:1224–1228, 2012.

[15] C. Currò, M. Sugiyama, H. Suzumura, and G. Valenti. Weak shock waves in isotropic solids at finite temperatures up to the melting point. *Continuum Mech. Thermodyn.*, 18:395–409, 2007.

[16] C. Currò, G. Valenti, M. Sugiyama, and S. Taniguchi. Propagation of an acceleration wave in layers of isotropic solids at finite temperatures. *Wave Motion*, 46:108–121, 2009.

[17] M. Fabrizio. A Cahn-Hilliard model for social integration. In J. J. Bissell, C. C. S. Caiado, S. E. Curtis, M. Goldstein, and B. Straughan, editors, *Modelling Social Problems and Health*. Wiley, 2013.

[18] M. Fabrizio, F. Franchi, and B. Straughan. On a model for thermoporoacoustic waves. *Int. J. Engng. Sci.*, 46:790–798, 2008.

[19] M. Fabrizio and A. Morro. *Electromagnetism of continuous media*. Oxford University Press, Oxford, 2003.

[20] R. A. Fisher. The wave of advance of advantageous genes. *Ann. Eugen.*, 7:355–369, 1937.

[21] M. Gentile and B. Straughan. Acceleration waves in double porosity elasticity. *Int. J. Engng. Sci.*, 73:10–16, 2013.

[22] M. Gentile and B. Straughan. Hyperbolic diffusion with Christov-Morro theory. *Mathematics and Computers in Simulation*, page doi: 10.1016/j.matcom.2012.07.010, 2013.

[23] A. E. Green and P. M. Naghdi. Thermoelasticity without energy-dissipation. *J. Elasticity*, 31:189–208, 1993.

[24] H. F. Huo and N. N. Song. Global stability for a binge drinking model with two stages. *Discrete Dynamics in Nature and Society*, 2012:829386, 2012.

[25] N. Hussaini and M. Winter. Travelling waves for an epidemic model with non-smooth treatment rates. *Journal of Statistical Mechanics - Theory and Experiment*, page P11019, 2010.

[26] P. M. Jordan. Growth and decay of acoustic acceleration waves in Darcy-type porous media. *Proc. Roy. Soc. London A*, 461:2749–2766, 2005.

[27] P. M. Jordan. Growth and decay of shock and acceleration waves in a traffic flow model with relaxation. *Physica D*, 207:220–229, 2005.

[28] P. M. Jordan. Finite amplitude acoustic travelling waves in a fluid that saturates a porous medium: Acceleration wave formation. *Phys. Letters A*, 355:216–221, 2006.

[29] P. M. Jordan. Growth, decay and bifurcation of shock amplitudes under the type-II flux law. *Proc. Roy. Soc. London A*, 463:2783–2798, 2007.

[30] P. M. Jordan. On the growth and decay of transverse acceleration waves on a nonlinear, externally damped string. *J. Sound and Vibration*, 311:597–607, 2008.

[31] P. M. Jordan. Some remarks on nonlinear poroacoustic phenomena. *Mathematics and Computers in Simulation*, 80:202–211, 2009.

[32] P. M. Jordan. A note on poroacoustic travelling waves under Forchheimer's law. *Phys. Letters A*, 377:1350–1357, 2013.

[33] P. M. Jordan and A. Puri. Qualitative results for solutions of the steady Fisher - KPP equation. *Appl. Math. Lett.*, 15:239–250, 2002.

[34] A. S. Kalula and F. Nyabadza. A theoretical model for substance abuse in the presence of treatment. *South African Journal of Science*, 108:96–107, 2012.

[35] A. Kandler and J. Steele. Ecological models of language competition. *Biological Theory*, 3:164–173, 2008.

[36] A. Kandler, R. Unger, and J. Steele. Language shift, bilingualism and the future of Britain's Celtic languages. *Phil. Trans. Royal Soc. London B*, 365:3855–3864, 2010.

[37] A. Kolmogoroff, I. Petrovsky, and N. Piscounoff. étude de l'équations de la diffusion avec croissance de la quantité de matière et son application a un prolème biologique. *Bull. Univ. Moskou, Ser. Int., Sec. A*, 1:1–25, 1937.

[38] J. Liu and T. Zhang. Global behaviour of a heroin epidemic model with distributed delays. *Appl. Math. Letters*, 24:1685–1692, 2011.

[39] A. Marasco. On the first-order speeds in any direction of acceleration waves in prestressed second - order isotropic, compressible, and homogeneous materials. *Mathematical and Computer Modelling*, 49:1644–1652, 2009.

[40] A. Marasco. Second - order effects on the wave propagation in elastic, isotropic, incompressible, and homogeneous media. *Int. J. Engng. Sci.*, 47:499–511, 2009.

[41] A. Marasco and A. Romano. On the acceleration waves in second - order elastic, isotropic, compressible, and homogeneous materials. *Mathematical and Computer Modelling*, 49:1504–1518, 2009.

[42] A. Morro. Evolution equations and thermodynamic restrictions for dissipative solids. *Math. Computer Modelling*, 52:1869–1876, 2010.

[43] G. Mulone and B. Straughan. A note on heroin epidemics. *Mathematical Biosciences*, 218:138–141, 2009.

[44] G. Mulone and B. Straughan. Modelling binge drinking. *International Journal of Biomathematics*, 5:1–14, 2012. Article No. 1250005.

[45] F. Nyabadza, S. Mukwembi, and B.G. Rodrigues. A graph theoretical perspective of a drug abuse epidemic model. *Physica A - Statistical Methods and Applications*, 390:1723–1732, 2011.

[46] F. Nyabadza, J. B. H. Njagarah, and R. J. Smith. Modelling the dynamics of crystal meth ("tik") abuse in the presence of drug-supply chains in South Africa. *Bull. Math. Biol.*, 75:24–48, 2013.

[47] P. Paoletti. Acceleration waves in complex materials. *Discrete and Continuous Dynamical Systems - B*, 17:637–659, 2012.

[48] T. Ruggeri and M. Sugiyama. Hyperbolicity, convexity and shock waves in one-dimensional crystalline solids. *J. Phys. A: Math. Gen.*, 38:4337–4347, 2005.

[49] V. D. Sharma and R. Venkatramani. Evolution of weak shocks in one dimensional planar and non - planar gas dynamics. *Int. J. Non-linear Mechanics*, 47:918–926, 2012.

[50] B. Straughan. *Stability, and wave motion in porous media*, volume 165 of *Appl. Math. Sci.* Springer, New York, 2008.

[51] B. Straughan. Acoustic waves in a Cattaneo-Christov gas. *Physics Letters A*, 374:2667–2669, 2010.

[52] B. Straughan. *Heat waves*, volume 177 of *Appl. Math. Sci.* Springer, New York, 2011.

[53] B. Straughan. Gene-culture shock waves. *Physics Letters A*, 377:2531–2534, 2013.

[54] B. Straughan. Stability and uniqueness in double porosity elasticity. *Int. J. Engng. Sci.*, 65:1–8, 2013.

[55] G. Valenti, C. Curro, and M. Sugiyama. Acceleration waves analysed by a new continuum model of solids incorporating microscopic thermal vibrations. *Continuum Mech. Thermodyn.*, 16:185–198, 2004.

[56] C. E. Walters, B. Straughan, and J. Kendal. Modelling alcohol problems: Total recovery. *Ricerche di Matematica*, 62:1–18, 2013. doi: 10.1007/s11587-012-0138-0.

[57] W. Wang, P. Fergola, S. Lombardo, and G. Mulone. Mathematical models of innovation diffusion with stage structure. *Applied Mathematical Modelling*, 30:129–146, 2006.